

The AdS particle

Subir Ghosh

Physics and Applied Mathematics Unit,
Indian Statistical Institute,
203 B. T. Road, Calcutta 700108, India

Abstract:

In this note we have considered a relativistic Nambu-Goto model for a particle in AdS metric. With appropriate gauge choice to fix the reparameterization invariance, we recover the previously discussed [7] "Exotic Oscillator". The Snyder algebra and subsequently the κ -Minkowski spacetime are also derived. Lastly we comment on the impossibility of constructing a noncommutative spacetime in the context of open string where only a curved target space is introduced.

Introduction: It is now accepted in the High Energy Physics community that nonlocality in quantum field theory, or in a more fundamental way the fuzziness (or Non-Commutativity (NC)) in space(time), will be an integral part of present-day theories. Intuitive arguments that are used in avoiding the paradoxes one faces in trying to localize a spacetime point within the Planck length [1] lead to a lower-bound in spacetime interval. This feature is also favored in the modifications of the Heisenberg uncertainty principle that one obtains in string scattering results (see for example [1]). It was first demonstrated by Snyder [2] that Lorentz invariance *and* discretization requires an NC spacetime.

The NC spacetime has been revived by the seminal work of Seiberg and Witten [3] who explicitly demonstrated the emergence of NC manifold in certain low energy limit of open strings moving in the background of a two form gauge field. In this instance, the NC spacetime is expressed by the Poisson bracket algebra (to be interpreted as commutators in the quantum analogue),

$$\{x^\mu, x^\nu\} = \theta^{\mu\nu}, \quad (1)$$

where $\theta^{\mu\nu}$ is a *c*-number constant. However, quantum field theories built on this spacetime do not enjoy Poincare invariance [4]. On the other hand, this type of pathology can be avoided if one works with NC spacetime of the Snyder form [2] or Lie algebraic form [5, 6]. In these examples the NC is *operatorial* in nature and thus it does not jeopardize the Lorentz invariance in relativistic models. The Lie algebra form of NC spacetime is typically given by,

$$\{x^\mu, x^\nu\} = C_\lambda^{\mu\nu} x^\lambda, \quad (2)$$

where the structure constants $C_\lambda^{\mu\nu}$ are constants.

In the present work, we will encounter both the Snyder [2] and Lie algebraic [5, 6] forms of NC. In particular, we will concentrate on a restricted class of Lie algebra valued spacetime known as κ -Minkowski spacetime (or κ -spacetime in short), that is described by the basic Poisson structure,

$$\{x_i, t\} = kx_i, \quad \{x_i, x_j\} = \{t, t\} = 0. \quad (3)$$

In the above, x_i and t denote the space and time operators respectively. The present work is in continuation of our recent paper [7].

Some of the important works in κ -spacetime that discusses, among other things, construction of a quantum field theory in κ -spacetime, are provided in [8, 9, 10]. Amelino-Camelia [11] has pioneered an alternative approach to quantum gravity - "the doubly special relativity" - in which *two* observer independent parameters, (the velocity of light and Planck's constant), are present. It has been shown [12] that κ -spacetime is a realization of the above. Furthermore, the mapping [12] between κ -spacetime and Snyder spacetime [2], (the first example of an NC spacetime), shows the inter-relation between these models and "two-time physics" [13], since the Snyder spacetime can be derived from two-time spaces in a particular gauge choice [14].

In [7] we have proposed a physically motivated realization of the κ -spacetime in a quantum mechanical model. This is quite in tune with the connection between the noncommutativity arising in the Landau problem and that in the open string boundary with a background field [1]. It is quite well known that for the planar, non-relativistic motion of a charged particle in a magnetic field (in the perpendicular direction), the particle configuration space becomes effectively noncommutative, if the dynamics is projected to the lowest Landau level. This is the celebrated Peirls substitution [15]. Physically this is applicable in the limit of strong magnetic

field [1]. However, it has gained significance in recent times because of its (qualitative) analogy with the noncommutativity in open string boundary manifolds (D -branes), in the presence of a background two form gauge field [3]. Unfortunately, a similar prototype of a simple physical system, picturizing the κ -spacetime was lacking. In our previous paper [7] we have shed some light on this area. Specifically, in [7], we have put forward a non-relativistic quantum mechanical model that has an underlying phase space algebra, isomorphic to the κ -Minkowski one (3). In [7] we have provided a Lagrangian of the model. As was mentioned in [7], (this point was noted in [14] as well), the action has an uncanny similarity with the structure of the dS or AdS metric.

Let us put the present work in its proper perspective. The κ -spacetime requires the time to be operatorial in nature since it bears a non-trivial commutation relation with the space variables as given in (3). However, our model in [7] was non-relativistic with conventional definition of time. To incorporate the operatorial behavior of time, we had to convert our model to a generalized one with reparameterization invariance [16] and then exploit this symmetry to (gauge)fix time accordingly so that the κ -spacetime algebra emerged. This somewhat round-about mechanism of [7] has led us to the present work where we extend the non-relativistic particle model of [7] to a relativistic, reparameterization invariant (Nambu-Goto) one. This allows us to fix the form of the time operator directly in the model. It is interesting to note that a similar type of time operator as in [7] reduces the present model to the one considered in [7]. We also recover a generalized form of the Snyder algebra [2], first given in [7]. But more importantly, now the AdS spacetime comes in to play directly and hence its connection to the κ -spacetime, via the Snyder algebra [2, 7] and exotic oscillator [7] becomes clear. The advantage of working in a gauge invariant framework is that other convenient gauge choices, besides the one mentioned above, are indeed possible.

In an interesting alternative approach, it might be possible to obtain the κ -spacetime directly from quantum (or noncommutative) AdS spacetime [17] ¹ One can obtain a broad indication of this connection from the fact that the classical AdS -space can be embedded in a higher dimensional space with two-time metric [13] and the κ -spacetime is directly related to the latter [12, 14]. At a more explicit level, since the κ -Poincare group can be obtained from the quantum AdS group by contraction [17], it is possible the corresponding spaces are related as well.

(Non-relativistic) Mechanical model for κ -spacetime: It will be worthwhile to recapitulate briefly the model proposed in [7]. We posited the Lagrangian,

$$L = \frac{m}{2} \vec{X}^2 - 2mkc\eta(\vec{X} \cdot \vec{X}) + c\eta^2 + 2mk^2c^2\eta^2\vec{X}^2, \quad (4)$$

where m denotes the mass of the non-relativistic particle and k and c are constant parameters, and as shown below, κ and c induce noncommutativity in phase space related to κ -spacetime.

In the Hamiltonian constraint analysis, as formulated by Dirac [18], with the canonical phase space,

$$\{X_i, P_j\} = \delta_{ij} \quad , \quad \{\eta, \pi\} = 1, \quad (5)$$

(where the sets (X_i, P_j) and (η, π) are decoupled), there are two Second Class Constraints

¹I thank Professor H. Steinacker for pointing this out.

(SCC) [18]²

$$\chi_1 \equiv \pi \quad , \quad \chi_2 \equiv \eta - k(\vec{P} \cdot \vec{X}) \quad . \quad (6)$$

Time independence of χ_1 reproduces χ_2 , ($\chi_2 = \{\chi_1, H\}$), with H representing the Hamiltonian. SCCs require the use of Dirac Brackets (DB) [18] defined by,

$$\{A, B\}_{DB} = \{A, B\} - \{A, \chi_i\} \{\chi_i, \chi_j\}^{-1} \{\chi_j, B\} \quad , \quad (7)$$

such that DB between an SCC and *any* operator vanishes. Note that $\{\chi_i, \chi_j\}^{-1}$ indicates inverse of the Poisson bracket matrix $\{\chi_i, \chi_j\}$. A brief computation [7] reveals the following non-canonical Dirac bracket algebra,

$$\{X_i, \eta\} = kX_i \quad , \quad \{P_i, \eta\} = -kP_i \quad , \quad \{X_i, P_j\} = \delta_{ij} \quad (8)$$

Since we will always deal with DBs the subscript DB is dropped. Clearly η behaves as time should in κ -spacetime, but a direct identification of η with time is obviously not possible. This was done in [7] by extending the model to a generally covariant. Incidentally, this way of exploiting a non-standard gauge condition to induce NC coordinates has been used in [19] in case of constant spacetime noncommutativity.

One can eliminate η and via an inverse Legendre transformation, obtain the following Lagrangian:

$$L = P_i \dot{X}_i - H = \frac{m}{2} [(\dot{X}_i)^2 - (2m\kappa^2 c) \frac{(X_i \dot{X}_i)^2}{1 + (2m\kappa^2 c) X_i^2}] \quad (9)$$

Depending on the sign of c , in the context of relativistic point particle to be demonstrated below, the above expression is generalized to dS or AdS spacetime.

Relativistic model for the AdS particle: The form of the non-relativistic action in (9) in some sense forces up on us its following relativistic counterpart:

$$L = -m[(\dot{X} \cdot \dot{X}) - \kappa \frac{(X \cdot \dot{X})^2}{1 + \kappa(X \cdot X)}]^{1/2} \equiv -mA, \quad (10)$$

where $(X \cdot \dot{X}) = X^\mu \dot{X}_\mu$ etc.. Here we have considered a generic form with a single parameter κ and $\dot{X}^\mu = \frac{dX^\mu}{d\tau}$. The above Nambu-Goto action clearly has the built-in AdS metric since the action is

$$\mathcal{A} = \int d\tau \sqrt{g_{\mu\nu} \dot{X}_\mu \dot{X}_\nu} \quad , \quad g_{\mu\nu} = \eta_{\mu\nu} - \frac{\kappa}{1 + \kappa X^\lambda X_\lambda} X^\mu X^\nu \quad (11)$$

The momentum is defined in the usual way,

$$P_\mu \equiv \frac{\delta L}{\delta \dot{X}^\mu} = -\frac{m}{A} [\dot{X}_\mu - \kappa \frac{(X \cdot \dot{X})}{1 + \kappa X^2} X_\mu] \quad (12)$$

We directly obtain a modified mass-shell condition

$$(P \cdot P) = m^2 - \kappa (P \cdot X)^2, \quad (13)$$

²In the Dirac terminology [18], First Class Constraints (FCC) commute with other constraints and generate gauge invariance. We will come across them in the present work later.

which reduces to the conventional one for $\kappa = 0$. The action has a τ -parameterization symmetry that generates a zero Hamiltonian:

$$H = (P.\dot{X}) - L = 0. \quad (14)$$

Note that the mass shell constraint in (13) represents the FCC [18]. We reexpress (4) in the form,

$$P_0 = \frac{1}{1 + \kappa X_0^2} [\kappa(\vec{P}.\vec{X})X_0 \pm m(1 + \kappa X_0^2)^{\frac{1}{2}} \{1 + \frac{\vec{P}^2}{m^2} - \frac{\kappa(\vec{P}.\vec{X})^2}{m^2(1 + \kappa X_0^2)}\}^{\frac{1}{2}}]. \quad (15)$$

We first demonstrate how the present system reduces to the non-relativistic model of [7]. Let us consider the large m or equivalently the non-relativistic limit,

$$P_0 \approx \frac{1}{1 + \kappa X_0^2} [\kappa(\vec{P}.\vec{X})X_0 \pm m(1 + \kappa X_0^2)^{\frac{1}{2}} \{1 + \frac{\vec{P}^2}{2m^2} - \frac{\kappa(\vec{P}.\vec{X})^2}{2m^2(1 + \kappa X_0^2)}\}]. \quad (16)$$

Keeping terms up to $O(\kappa)$ we rewrite P_0 in the following suggestive way,

$$P_0 \approx m + \frac{\vec{P}^2}{2m} - \frac{\kappa}{2m}(\vec{P}.\vec{X})^2 + \kappa X_0[(\vec{P}.\vec{X}) - \frac{m}{2}X_0(1 + \frac{\vec{P}^2}{2m^2})]. \quad (17)$$

Thus, modulo the last term, we have obtained the expression for the Hamiltonian derived in [7]. We can now exploit the reparameterization symmetry to introduce the gauge condition,

$$X_0 = \frac{2(\vec{P}.\vec{X})}{m}(1 + \frac{\vec{P}^2}{2m^2})^{-1}. \quad (18)$$

Clearly the gauge fixed Hamiltonian reduces to that of [7].

However, the gauge constraint has rendered the FCC system to an SCC one with the SCC pair,

$$\phi_1 \equiv P_0 - \frac{\vec{P}^2}{2m} + \frac{\kappa}{2m}(\vec{P}.\vec{X})^2 + O(\frac{1}{m^3}), \quad \phi_2 \equiv X_0 - \frac{2(\vec{P}.\vec{X})}{m} + O(\frac{1}{m^3}). \quad (19)$$

They satisfy a non-zero Poisson bracket:

$$\{\phi_1, \phi_2\} = (1 - \frac{2\vec{P}^2}{m^2}) \equiv \alpha. \quad (20)$$

The canonical phase space with $\{P_\mu, X^\nu\} = \eta_\mu^\nu$ gets modified to the Dirac brackets,

$$\begin{aligned} \{X_i, X_j\} &= \frac{2}{\alpha m^2}(X_i P_j - X_j P_i) \quad , \quad \{P_i, P_j\} = 0 \\ \{X_i, P_j\} &= \delta_{ij} + \frac{2}{\alpha m^2}P_i P_j. \end{aligned} \quad (21)$$

The Dirac brackets with time operator X^0 turn out to be,

$$\{X_i, X_0\} = -\frac{2X_i}{\alpha m} \quad , \quad \{P_i, X_0\} = \frac{2P_i}{\alpha m}. \quad (22)$$

Time evolution is given by the Heisenberg equations of motion,

$$\dot{X}_i = \{X_i, P_0\} = \frac{1}{m}(P_i - \kappa(\vec{P} \cdot \vec{X})X_i), \quad \dot{P}_i = \{P_i, P_0\} = \frac{\kappa}{m\alpha}(\vec{P} \cdot \vec{X})P_i. \quad (23)$$

A further iteration reveals the dynamics:

$$\ddot{X}_i = -\frac{2\kappa}{m}P_0X_i. \quad (24)$$

The other equation for \ddot{P}_i is given below,

$$\ddot{P}_i = \frac{2\kappa}{m}\left(\frac{\vec{P}^2}{2m} + \frac{\kappa}{2m}(\vec{P} \cdot \vec{X})^2\right)P_i. \quad (25)$$

Thus we have recovered the "Exotic Oscillator" dynamics of [7]. A redefinition of the variables, as given in [7], will lead to the κ -spacetime. The generalized form of the Snyder algebra, first given [7], is also recovered here in (21). Notice that in the approximations that we have considered, the NC algebra is κ -independent but κ appears in the dynamics because otherwise we will have a free particle system.

As we are interested in the κ -spacetime, quite obviously the choice of time (X_0) that is obtained from the form of gauge fixing is not canonical. Hence it might be interesting to compare the dynamics with this choice of time and the conventional (c -number parameter) one $X_0 = \tau$ by considering an alternative choice of gauge gauge $\phi_2 \approx X_0 - \tau$. In this case, the SCC system is,

$$\begin{aligned} \phi_1 &\approx P_0 - \left[\frac{\vec{P}^2}{2m} - \frac{\kappa}{2m}(\vec{P} \cdot \vec{X})^2 + \kappa\tau[(\vec{P} \cdot \vec{X}) - \frac{m}{2}\tau(1 + \frac{\vec{P}^2}{2m^2})]\right], \\ \phi_2 &\approx X_0 - \tau, \end{aligned} \quad (26)$$

where ϕ_2 has been used in ϕ_1 . Since now the Hamiltonian P_0 depends explicitly on time τ , one has to consider the generalized form of Heisenberg equation for a generic operator A ,

$$\frac{dA}{d\tau} = \frac{\partial A}{\partial \tau} + \{A, P_0\}. \quad (27)$$

It is clear that the canonical structure ($\{X_i, P_j\} = \delta_{ij}$) of phase space is not altered by this gauge choice. Thus in case of conventional time, the dynamics is governed by,

$$\ddot{X}_i = -\frac{2\kappa}{m}P_0X_i + \kappa(X_i - \frac{\tau P_i}{m}). \quad (28)$$

We find that the basic characteristics of the dynamics of the "Exotic Oscillator" obtained in (24) remains intact, since vanishing of the last term defines the constant non-relativistic momentum. Perhaps this feature is not so surprising if we recall that in [7] the "Exotic Oscillator" dynamics was reproduced in conventional time with canonical phase space brackets.

Open String in curved background: The next step in generalization ought to be the de Sitter string that is string moving in a de Sitter background. However, instead of considering de Sitter metric in particular, we will consider a generic form of X^μ -dependent metric $G_{\mu\nu}(X)$. In

a previous work [20] we have shown how the boundary conditions affect the Poisson bracket structures, considering the specific case of spacetime noncommutativity arising from the non-trivial boundary conditions occurring in the interacting system of open string and two-form background gauge field. As a concrete example, in [20] we have shown the noncommutativity appearing in the open string boundary manifolds (D -branes) in the presence of a two-form background field can be rigorously obtained once the boundary conditions are properly taken in to account. Here we will show that a *curved* metric indeed modifies the boundary conditions but *it does not induce noncommutativity*.

The canonical phase space algebra

$$\{X^\mu(\sigma), X^\nu(\sigma')\} = \{\Pi_\mu(\sigma), \Pi_\nu(\sigma')\} = 0, \quad \{\Pi_\mu(\sigma), X^\nu(\sigma')\} = g_\mu^\nu \delta(\sigma - \sigma'),$$

is incompatible with the boundary conditions that one obtains for free open strings at the boundary and a modified form of δ -function ($\Delta(\sigma - \sigma')$) appears, whose σ -derivative vanished at the string boundary [21]. On the other hand, for open strings moving in the presence of two-form background field, the modified boundary conditions require a non-vanishing $\{X_\mu(\sigma), X_\nu(\sigma')\}$, indicating noncommutativity [20]. This point is explained at the end.

The Polyakov action for the motion of an open string in a curved background $G_{\mu\nu}(X)$ is,

$$\mathcal{S} = -\frac{1}{2} \int d\sigma d\tau \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}, \quad (29)$$

where $G_{\mu\nu}$ is the curved target space metric and g_{ab} is the induced metric. The momentum is defined below:

$$\Pi_\mu = \frac{\delta \mathcal{S}}{\delta \partial_0 X^\mu} = -\sqrt{-g} G_{\mu\nu} g^{0a} \partial_a X^\nu = -\sqrt{-g} G_{\mu\nu} \partial^0 X^\nu \quad (30)$$

Variation of the induced metric g^{ab} determines the energy-momentum tensor

$$T_{ab} = (-\partial_a X^\mu \partial_b X^\nu + \frac{1}{2} g_{ab} g^{cd} \partial_c X^\mu \partial_d X^\nu) G_{\mu\nu}. \quad (31)$$

Vanishing of the above,

$$T_{ab} = 0 \quad (32)$$

provides the constraints of the theory that confirms reparameterization invariance. From the Hamiltonian constraint analysis point of view, the following combinations of constraints are useful:

$$\begin{aligned} T_{11} &= \frac{1}{2} (g \partial^0 X^\mu \partial^0 X^\nu - \partial_1 X^\mu \partial_1 X^\nu) G_{\mu\nu}, \\ \sqrt{-g} T^0_1 &= -\sqrt{-g} \partial^0 X^\mu \partial_1 X^\nu G_{\mu\nu}. \end{aligned} \quad (33)$$

We can reexpress the constraints in terms of phase space variables,

$$\begin{aligned} T_{11} \equiv \chi_1 &= -\frac{1}{2} (\Pi_\mu \Pi_\nu G^{\mu\nu} + \partial_1 X^\mu \partial_1 X^\nu G_{\mu\nu}), \\ \sqrt{-g} T^0_1 \equiv \chi_2 &= \Pi_\mu \partial_1 X^\mu. \end{aligned} \quad (34)$$

The constraints χ_1 and χ_2 are FCC [18] and satisfy the normal diffeomorphism algebra:

$$\{\psi_1(\sigma), \psi_1(\sigma')\} = 4(\psi_2(\sigma) + \psi_2(\sigma')) \partial_\sigma \delta(\sigma - \sigma'),$$

$$\begin{aligned}\{\psi_2(\sigma), \psi_1(\sigma')\} &= (\psi_1(\sigma) + \psi_1(\sigma'))\partial_\sigma\delta(\sigma - \sigma'), \\ \{\psi_2(\sigma), \psi_2(\sigma')\} &= (\psi_2(\sigma) + \psi_2(\sigma'))\partial_\sigma\delta(\sigma - \sigma').\end{aligned}\tag{35}$$

Let us now return to the Lagrangian framework. The equation of motion and boundary condition arising from the action (29) are respectively,

$$\partial_b[\sqrt{-g}g^{ab}\partial_a X^\mu G_{\mu\nu}] - \frac{1}{2}\sqrt{-g}g^{ab}\partial_a X^\mu\partial_b X^\lambda\frac{\delta G_{\mu\lambda}}{\delta X^\nu} = 0,\tag{36}$$

$$\sqrt{-g}g^{1a}(\partial_a X^\mu)G_{\mu\nu} \big|_{\sigma=0,\pi} = 0,\tag{37}$$

where $\sigma = 0, \pi$ are the string extremities. The boundary condition, expressed in terms of phase space variables, becomes,

$$\partial_1 X^\mu + \sqrt{-g}g^{10}\Pi^\mu \big|_{\sigma=0,\pi} = 0,\tag{38}$$

where some unimportant factors have been dropped. Notice that the diffeomorphism algebra (35) ensures that we can choose a gauge, in particular the conformal gauge, in which case $g^{10} = 0$, and we are left with $\partial_1 X^\mu \big|_{\sigma=0,\pi} = 0$ as the boundary condition. This boundary condition is compatible with commutative spacetime. Comparing with our earlier work [20] we establish that spacetime noncommutativity is not induced by only considering a curved spacetime.

Let us briefly elaborate on the last comment regarding [20] and its connection to the present conclusion. The importance of obtaining the purported noncommutativity from different (in particular Hamiltonian) formalisms was stressed in the original work of Seiberg and Witten [3], since the concept of noncommutative spacetime was quite alien to the physics community. The first works in this connection [22], tried to establish that the noncommutative spacetime algebra should be interpreted as Dirac brackets [18] provided one treats *the boundary conditions as Second Class Constraints* [18]. However, these works [22] contained various assumptions and computational steps that were ambiguous from the perspective of conventional Dirac analysis [18], [21] of constrained systems. Subsequently it was realized [20], [23] that the problem lies at the basic premises of [22]: The boundary conditions are *not* to be treated as (field theoretic) constraints since the former apply *only* at the boundaries whereas the latter are valid for the *whole* region of phase space. This led us to our analysis [20] where we generalized the earlier works [21]. It was demonstrated in [21] that for the case of open strings, basic phase space Poisson brackets are to be modified in order to be consistent with boundary conditions. In [20] we explicitly showed that the boundary conditions for the interacting system of open string in an external two-form gauge field are consistent only with a noncommutative spacetime algebra. The counter intuitive idea of interpreting boundary conditions as constraints as in [22] need not be introduced at all. This explains our conclusion that in the present case that the boundary conditions do not require a noncommutative spacetime.

References

- [1] For a review, see R.Szabo, Int.J.Mod.Phys. A19 (2004)1837.
- [2] H.S.Snyder, Phys.Rev. 71 (1947)38.

- [3] N.Seiberg and E.Witten, JHEP 9909(1999)032. For reviews see for example M.R.Douglas and N.A.Nekrasov, Rev.Mod.Phys. 73(2001)977; R.J.Szabo, *Quantum Field Theory on Noncommutative Spaces*, hep-th/0109162.
- [4] S.Ghosh, Mod.Phys.Lett.A19 (2004)2505.
- [5] S.Doplicher, K.Fredenhagen and J.E.Roberts, Phys.Lett. B331 39(1994).
- [6] J.Madore, S.Schraml, P.Schupp and J.Wess, Eur.Phys.J. C16 (2000) 161; G.Amelino-Camelia, *Field Theories on Canonical and Lie-Algebra Noncommutative Spacetimes*, hep-th/0205047.
- [7] S.Ghosh and Probir Pal, Phys.Lett.B 618 (2005)243, (hep-th/0502192).
- [8] J. Lukierski, H. Ruegg, W.J. Zakrzewski, Annals Phys. 243 (1995)90; P. Kosiński, J. Lukierski, P. Maślanka, Phys.Rev. D62 (2000) 025004.
- [9] G.Amelino-Camelia and M.Arzano, Phys.Rev. D65 (2002)084044; A.Agostini, G.Amelino-Camelia and F.D’Andrea, Int.J.Mod.Phys. A19 (2004)5187.
- [10] M.Dimitrijevic, L.Jonke, L.Moeller, E.Tsouchnika, J.Wess and M.Wohlgenannt, Czech.J.Phys. 54 (2004)1243; Eur.Phys.J. C31 (2003)129. 11
- [11] G.Amelino-Camelia, Nature 418 (2002)34, gr-qc/0207049.
- [12] J.Kowalski-Glikman and S.Nowak, Int.J.Mod.Phys. D12 (2003)299.
- [13] I.Bars, *Two-time physics*, hep-th/9809034.
- [14] J.M.Romero and A.Zamora, Phys.Rev. D70 (2004)105006.
- [15] R.Peirls, Z.Phys. 80 (1933)763.
- [16] For a lucid discussion see M.Henneaux and C.Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, 1991.
- [17] H.Steinacker, Adv.Theor.Math.Phys. 4 (2000)155 (hep-th/9910037); *Quantum Anti de Sitter space and sphere at roots of unity*, hep-th/9910037.
- [18] P.A.M.Dirac, *Lectures on Quantum Mechanics*, Yeshiva University Press, New York, 1964.
- [19] A.Pinzul and A.Stern, Phys.Lett. B593 (2004)279; R.Banerjee, B.Chakraborty and S. Gangopadhyay, J.Phys.A:Math.Gen. 38 (2005)957 (hep-th/0405178).
- [20] R.Banerjee, B.Chakraborty and S.Ghosh, Phys. Lett. B537 (2002)340.
- [21] A.J.Hanson, T.Regge and C.Teitelboim, *Constrained Hamiltonian System*, Roma, Accademia Nazionale Dei Lincei, (1976).

- [22] F.Ardalan, H.Arfaei and M.M.Sheikh-Jabbari, JHEP 9902 016(1999); W.T.Kim and J.J.Oh, Mod.Phys.Lett. A15 1597(2000); C.-S.Chu and P.M.Ho, Nucl.Phys. B568 (2000)447; M.M.Sheikh-Jabbari and A.Shirzad, Eur.Phys.J.C. 19 (2001)383; K.-I.Tezuka, hep-th/0201171.
- [23] F.Loran, Phys.Lett. B544 (2002)199, W.He and L.Zhao, Phys.Lett. B532 (2002)345.